Exercise 40

By comparing Definitions 2, 3, and 4, prove Theorem 2.3.1.

Solution

Theorem 2.3.1 is on page 99 and says that

$$\lim_{x \to a} f(x) = L \qquad \text{if and only if} \qquad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x).$$

Suppose first that

$$\lim_{x \to a} f(x) = L.$$

According to Definition 2, this is logically equivalent to

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon \tag{1}$$

for all positive ε. Notice that the hypothesis can be rewritten as

$$|x - a| < \delta$$

$$-\delta < x - a < \delta$$

$$a - \delta < x < a + \delta.$$

This interval is the union of two smaller intervals, $a - \delta < x < a$ and $a < x < a + \delta$. As a result, (1) becomes

$$\text{if} \quad (a - \delta < x < a \quad \text{or} \quad a < x < a + \delta) \qquad \text{then} \qquad |f(x) - L| < \varepsilon.$$

Break up this if-then statement into two.

$$\begin{cases} \text{if} \quad a - \delta < x < a & \text{then} \qquad |f(x) - L| < \varepsilon \\ \text{if} \quad a < x < a + \delta & \text{then} \qquad |f(x) - L| < \varepsilon \end{cases}$$

Therefore, by Definition 3 and Definition 4, respectively,

$$\begin{cases} \lim_{x \to a^-} f(x) = L \\ \lim_{x \to a^+} f(x) = L \end{cases}.$$

Suppose secondly that

$$\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x).$$

Then

$$\begin{cases} \displaystyle\lim_{x \to a^-} f(x) = L \\ \displaystyle\lim_{x \to a^+} f(x) = L \end{cases}.$$

According to Definition 3 and Definition 4, these limits are logically equivalent to

$$\begin{cases} \text{if} \quad a - \delta < x < a \qquad \text{then} \qquad |f(x) - L| < \varepsilon \\ \text{if} \quad a < x < a + \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon \end{cases},$$

respectively. Combine these if-then statements.

$$\text{if} \quad (a - \delta < x < a \quad \text{or} \quad a < x < a + \delta) \qquad \text{then} \qquad |f(x) - L| < \varepsilon. \tag{2}$$

The union of these two intervals, $a - \delta < x < a$ and $a < x < a + \delta$, is

$$a - \delta < x < a + \delta$$

$$-\delta < x - a < \delta$$

$$|x - a| < \delta.$$

Consequently, (2) becomes

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon.$$

By Definition 2, this is logically equivalent to

$$\lim_{x \to a} f(x) = L.$$

Theorem 2.3.1 is proven.